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# Commuting Projections on Graphs

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Numerical Linear Algebra with Applications

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# COMMUTING PROJECTIONS ON GRAPHS

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ABSTRACT. For a given (connected) graph, we consider vector spaces of (discrete) functions defined on its vertices and its edges. These two spaces are related by a discrete gradient operator,  $\text{Grad}$  and its adjoint,  $-\text{Div}$ , referred to as (negative) discrete divergence. We also consider a coarse graph obtained by aggregation of vertices of the original one. Then a coarse vertex space is identified with the subspace of piecewise constant functions over the aggregates. We consider the  $\ell_2$ -projection  $Q_H$  onto the space of these piecewise constants. In the present paper, our main result is the construction of a projection  $\pi_H$  from the original edge-space onto a properly constructed coarse edge-space associated with the edges of the coarse graph. The projections  $\pi_H$  and  $Q_H$  commute with the discrete divergence operator, i.e., we have  $\text{div } \pi_H = Q_H \text{ div}$ . The respective pair of coarse edge-space and coarse vertex-space offer the potential to construct two-level, and by recursion, multilevel methods for the mixed formulation of the graph Laplacian which utilizes the discrete divergence operator. The performance of one two-level method with overlapping Schwarz smoothing and correction based on the constructed coarse spaces for solving such mixed graph Laplacian systems is illustrated on a number of graph examples.

## 1. INTRODUCTION

This paper considers discrete divergence operator  $\text{Div}$  on functions defined on graph edges as the negative adjoint of the (discrete) gradient operator  $\text{Grad}$ . The latter acts on functions defined on graph vertices and takes values on the graph edges. The commuting diagram property which we prove here is similar to the one introduced in [1] for finite element spaces and agglomerated coarse spaces (finite element graphs). The  $\text{Grad}$  operator naturally appears in the factorization of the popular graph Laplacian operator (cf. [2, 3, 4]) and also in the finite difference discretizations of elliptic equations on unstructured grids [5]. For solving problems with graph Laplacian, which gives rise to symmetric positive semi-definite  $M$ -matrices, multilevel methods such as algebraic multigrid methods (or AMG, see [6]) are natural candidates. For example, coarsening methods based on aggregation and associated coarse spaces of piecewise constant functions have been successfully used in [7], [8], and [9]. The choice of coarse space with piecewise constants is natural also for matrices coming from cell-centered finite difference and interior penalty (discontinuous Galerkin) discretizations of second order elliptic PDEs, which motivated by the fact that the space for the pressure unknown in the mixed formulation of second order elliptic equations can naturally be chosen discontinuous (piecewise constants, in the lowest order case).

In this paper, we are interested in graph coarsening by aggregation which seems to be the method of choice in practice. Namely, by grouping the vertices of the original graph into sets of connected vertices that form a non-overlapping partitioning, we define a coarse graph

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with coarse vertices being the aggregates. We also define edges of the coarse graph; namely, a pair of aggregates are connected by a coarse edge if there is an edge of the original graph with its two different vertices belonging to different aggregates of this pair. With the coarse graph, we associate a coarse space of function of piecewise constants,  $S_H$ , with respect to the set of aggregates that define the coarse graph vertices. The main result of this paper is the construction of a projection operator  $\pi_H$  acting on the space of functions  $\mathbf{U}$  defined on the edges of a given graph with its range being an appropriately constructed coarse-edge space  $\mathbf{U}_H$ , a subspace of  $\mathbf{U}$ . This projection operator, together with the piecewise constant projection  $Q_H$  acting on the vertex-based space  $S$  (on the same graph) and range  $S_H$ , define respective pair of coarse spaces that can be used to define coarse versions of the discrete Laplacian in mixed setting (to be introduced later on). A main property is that the pair of projections  $\pi_H$ ,  $Q_H$  commute with the discrete divergence operator, namely, we have

$$\text{Div } \pi_H = Q_H \text{Div}.$$

This property and the fact that the coarse basis  $\{\varphi_I\}$  of  $\mathbf{U}_H \subset \mathbf{U}$ , that we construct has certain energy minimizing property is proven important in the construction of multilevel methods for mixed systems involving the (continuous PDE) divergence operator (cf. [1], [10]). The same applies to the mixed system associated with the graph Laplacian. Additionally, based on its better energy minimizing property (by construction)  $\mathbf{U}_H$  can be used for constructing multilevel methods for graph Laplacian defined on a graph that has as vertices the edges of the original graph. In the present paper, however, our focus is to introduce the spaces and operators and study their properties, including their performance in two-level methods. Their possible further application to formulate and solve optimization problems on edges or the pair of edges and vertices, will be studied elsewhere.

The remainder of this paper is structured as follows. In Section 2, we introduce the basic definitions, and study some main properties of the discrete gradient and divergence operators. In Section 3, the main construction of the local basis  $\{\varphi_I\}$  of the coarse edge space  $\mathbf{U}_H$  is introduced. Its energy minimizing property is studied in Section 4. The main commutativity property of the projections  $Q_H$  and  $\pi_H$  is proven in Section 5. Some illustration of the constructed subgraphs and numerical verification of some of the theoretical results is shown in Section 6.

## 2. NOTATION AND PRELIMINARIES

Let  $G$  be a graph with set of vertices  $\mathcal{V} = \{1, \dots, n\}$  and set of edges  $\mathcal{E}$ . We assume that  $G$  is undirected, that is if  $(i, j) \in \mathcal{E}$  then  $(j, i) \in \mathcal{E}$ . For weighted graph  $G$ , to each edge  $e \in \mathcal{E}$ , we assign a weight  $w(e)$  which is a real number. Typically,  $w(e) > 0$ . For undirected weighted graph  $G$ , we assume that  $w(e) = w(e')$  for  $e = (i, j)$  and  $e' = (j, i)$ . In what follows, we can either assume that  $\mathcal{E}$  contains only one of the edges  $e = (i, j)$  and  $e' = (j, i)$ , or all functions  $\psi$  defined on  $\mathcal{E}$ , have the property  $\psi(e) = \psi(e')$ . The latter assumption is a bit odd especially when we define inner products and operators on functions defined on  $\mathcal{E}$ , but works just fine.

In the present paper, we deal with subgraphs of a given  $G$  and with vector spaces and operators associated with them. The corresponding definitions and constructions are found next.

**2.1. Basic definitions.** We now introduce several notions which we need later on. With the graph  $G = (\mathcal{V}, \mathcal{E})$ , we associate two vector spaces  $S = \mathbb{R}^n$  and  $\mathbf{U} = \mathbb{R}^{n_{\mathcal{E}}}$ . The inner

product and respective norms in these spaces are denoted by  $(\cdot, \cdot)$  and  $\|\cdot\|$  respectively. We use the same notation for the inner product and the norm for both  $S$  and  $\mathbf{U}$ . This should not cause confusion, since we use bold letters to indicate elements of  $\mathbf{U}$ . Thus, we have,

$$\begin{aligned} (p, q) &= \sum_{i=1}^n p_i q_i = \sum_{j \in \mathcal{V}} p_j q_j, \quad \|p\|^2 = (p, p), \quad \text{for all } p \in S, q \in S, \\ (\mathbf{v}, \mathbf{w}) &= \sum_{k=1}^{n_{\mathcal{E}}} \mathbf{v}_k \mathbf{w}_k = \sum_{e \in \mathcal{E}} \mathbf{v}_e \mathbf{w}_e. \quad \|\mathbf{v}\|^2 = (\mathbf{v}, \mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbf{U}, \mathbf{w} \in \mathbf{U}. \end{aligned}$$

The cardinality of a finite set  $\Omega$  is denoted by  $|\Omega|$ . For example, we have  $n = |\mathcal{V}|$ ,  $n_{\mathcal{E}} = |\mathcal{E}|$ , etc. We call a graph  $G = (\mathcal{V}, \mathcal{E})$  *connected* if for every pair of  $i \in \mathcal{V}$  and  $j \in \mathcal{V}$  there exist  $m+1$ -vertices  $\{k_{\ell}\}_{\ell=0}^m \subset \mathcal{V}$  with  $i = k_0$ , and  $j = k_m$ , and such that  $(k_{\ell-1}, k_{\ell}) \in \mathcal{E}$  for all  $\ell = 1, \dots, m$ . In other words, there is a path between  $i$  and  $j$  formed by the edges of  $G$ .

A sub-graph of  $G$ ,  $\mathbf{a} \subset G$ , is defined as  $\mathbf{a} = (\mathcal{V}_{\mathbf{a}}, \mathcal{E}_{\mathbf{a}})$ , and is a graph whose vertices  $\mathcal{V}_{\mathbf{a}}$  are subset of  $\mathcal{V}$  and its set of edges  $\mathcal{E}_{\mathbf{a}} \subset \mathcal{E}$  is the set of edges from  $\mathcal{E}$  which have *both* their ends in  $\mathcal{V}_{\mathbf{a}}$ . If  $\mathbf{a}$  is itself a connected graph, we refer to  $\mathbf{a}$  as *aggregate*. Note that once the subset of vertices  $\mathcal{V}_{\mathbf{a}}$  is fixed the subgraph  $\mathbf{a}$  is uniquely determined.

Consider now the set of edges  $\mathcal{I}_{\mathbf{a}} \subset \mathcal{E}$  which have exactly one end in  $\mathcal{V}_{\mathbf{a}}$ . The set of vertices that are end points of edges in  $\mathcal{I}_{\mathbf{a}}$  and are also in  $\mathbf{a}$  are called *boundary vertices* of  $\mathbf{a}$  and we denote this set by  $\mathcal{V}_{\partial \mathbf{a}}$ . Clearly  $\mathcal{V}_{\partial \mathbf{a}} \subset \mathcal{V}_{\mathbf{a}} \subset \mathcal{V}$ . Edges from  $\mathcal{E}_{\mathbf{a}}$  which have *both* their ends in  $\mathcal{V}_{\partial \mathbf{a}}$  form  $\mathcal{E}_{\partial \mathbf{a}}$ . Finally, the subgraph  $\partial \mathbf{a} = (\mathcal{V}_{\partial \mathbf{a}}, \mathcal{E}_{\partial \mathbf{a}})$  is called the boundary of  $\mathbf{a}$ . We assume that  $|\mathcal{V}_{\mathbf{a}}| > 1$  for the sub-graphs (aggregates) that we consider later on. In such a case,  $\mathcal{E}_{\mathbf{a}}$  contains at least one edge, whereas we note that  $\mathcal{E}_{\partial \mathbf{a}}$  could be empty.

**2.2. Weighted graph Laplacian, discrete divergence and gradient.** The discrete gradient  $\text{Grad} : S \mapsto \mathbf{U}$  is defined as follows

$$(2.1) \quad (\text{Grad } q)_e = \epsilon_{ij}(q_i - q_j).$$

The sign  $\epsilon_{ij} = \pm 1$ , satisfies  $\epsilon_{ij} = -\epsilon_{ji}$  is chosen a priori and fixed. The considerations that follow do not depend on what choice of sign we made, as long as  $\epsilon_{ij} = -\epsilon_{ji}$ . If one wants to remove this little ambiguity, a choice  $\epsilon_{ij} = \text{sign}(i - j)$  works. Next, following some traditional notation, we define  $\text{Div} : \mathbf{U} \mapsto S$  as the adjoint to  $(-\text{Grad})$ , i.e.,

$$(2.2) \quad (\text{Div } \mathbf{v}, q) = -(\mathbf{v}, \text{Grad } q), \quad \text{for all } q \in S.$$

Given a positive function  $w : \mathcal{E} \mapsto \mathbb{R}$ , which assigns a weight to each edge, the (weighted) graph Laplacian  $\mathcal{L}$  is defined via the following quadratic form  $(\mathcal{L} \cdot, \cdot) : S \times S \mapsto \mathbb{R}$ ,

$$(2.3) \quad (\mathcal{L}p, q) = \sum_{e=(i,j) \in \mathcal{E}} w(e)(p_i - p_j)(q_i - q_j) = (W \text{Grad } p, \text{Grad } q).$$

Here  $W : \mathbf{U} \mapsto \mathbf{U}$  is a diagonal matrix such that if  $W = (W_{e,e'})$ , then  $W_{e,e'} = w(e)\delta_{e,e'}$  (and  $\delta_{e,e'}$  is the Kronecker symbol). Using the divergence operator defined in (2.2), we have the factorization

$$(\mathcal{L}p, q) = (W \text{Grad } p, \text{Grad } q) = -(\text{Div}(W \text{Grad } p), q).$$

That is, we have  $\mathcal{L} = -\text{Div } W \text{Grad}$ . Using this factorization, given a problem  $\mathcal{L}p = f$ , it can be posed as the following mixed system for  $\mathbf{u} \in \mathbf{U}$  and  $p \in S$ ,

$$(2.4) \quad \begin{aligned} (W^{-1}\mathbf{u}, \mathbf{v}) + (\text{Div } \mathbf{v}, p) &= 0, \quad \text{for all } \mathbf{v} \in \mathbf{U}, \\ (\text{Div } \mathbf{u}, q) &= (-f, q), \quad \text{for all } q \in S. \end{aligned}$$

We note that the operators  $\text{Grad}$ ,  $\text{Div}$ ,  $W$  and  $\mathcal{L}$  depend on  $G$ , and we write  $\text{Grad}_G$ ,  $\text{Div}_G$ ,  $W_G$  and  $\mathcal{L}_G$  when such dependence needs to be emphasized.

**2.3. Subspaces and projections.** For an integer  $N > 0$ , for  $\Omega \subset \{1, \dots, N\}$ , we define the vector spaces  $S = \mathbb{R}^N$  and  $S_\Omega \subset S$ :

$$S_\Omega = \{v \in S, \text{ such that } v_i = 0 \text{ for all } i \notin \Omega\}.$$

The  $\ell_2$ -orthogonal projection on  $S_\Omega$ , denoted by  $Q_\Omega$  is defined in the usual way:

For  $q \in S$ ,  $Q_\Omega q \in S_\Omega$  is the vector that is the same as  $q$  on  $\Omega$  and zero otherwise, that is,

$$(2.5) \quad (Q_\Omega q)_i = \begin{cases} q_i, & \text{for all } i \in \Omega, \\ 0, & \text{for all } i \notin \Omega. \end{cases}$$

Equivalently, one can verify the following properties of  $Q = Q_\Omega$  (often used in the proofs) which hold for all  $p \in S$ ,  $q \in S$  and  $s \in S_\Omega$ :

$$(Qq, s) = (q, s), \quad Q^2 = Q, \quad (Qp, q) = (p, Qq), \quad (Qp, Qq) = (p, Qq).$$

We now introduce the “constant vector”  $\mathbb{1} \in \mathbb{R}^N$ ,  $\mathbb{1}_i = 1$ , for  $i = 1, \dots, N$ . We write  $\mathbb{1}_\Omega = Q_\Omega \mathbb{1}$  and, we also denote the  $\ell_2$ -based projection on the one-dimensional space spanned by  $\mathbb{1}_\Omega$  with  $Q_{0,\Omega}$ . If  $\langle q \rangle_\Omega$  is the  $\ell_2$ -weighted average value of  $q$  on  $\Omega$ , i.e.,

$$\langle q \rangle_\Omega = \frac{(\mathbb{1}_\Omega, q)}{(\mathbb{1}_\Omega, \mathbb{1}_\Omega)},$$

then have the following representation

$$Q_{0,\Omega} q = \langle q \rangle_\Omega \mathbb{1}_\Omega.$$

In the special case, when  $\Omega = \{1, \dots, N\}$  (i.e.  $S_\Omega = S = \mathbb{R}^N$ ) we write  $Q_0$ , instead of  $Q_{0,\Omega}$ .

We use these definitions with  $N = n$ ,  $N = n_\mathcal{E}$ ,  $\Omega = \mathcal{V}_\mathbf{a}$ ,  $\Omega = \mathcal{E}_\mathbf{a}$ , or  $\Omega = \mathcal{I}_\mathbf{a}$ . In the case when  $\Omega = \mathcal{V}_\mathbf{a}$  for an aggregate  $\mathbf{a}$ , we denote the vector subspace by  $S_\mathbf{a}$ , and the corresponding projections are denoted by  $Q_\mathbf{a}$ , or  $Q_{0,\mathbf{a}}$  respectively. We use bold letters for the spaces and projections on edges. Thus, when  $\Omega = \mathcal{E}_\mathbf{a}$ , we denote the vector subspace by  $\mathbf{U}_\mathbf{a}$ , or, in general, when  $\Omega = \mathcal{I} \subset \mathcal{E}$  we write  $\mathbf{U}_\mathcal{I}$  for the corresponding vector space. The projection on  $\mathbf{U}_\mathcal{I}$ , is denoted accordingly by  $\mathbf{Q}_\mathcal{I}$ .

**2.4. A main property of Div operator.** The next proposition is an important ingredient that shows that the saddle point problems defined later are uniquely solvable.

**Proposition 2.1.** *Let  $X = (\mathcal{V}_X, \mathcal{E}_X)$ , with  $|\mathcal{V}_X| = n_X$  and  $|\mathcal{E}| = m_X$  be a connected graph and  $W_X : \mathbb{R}^{m_X} \mapsto \mathbb{R}^{m_X}$  be a diagonal matrix of edge weights for  $X$ . We assume that  $(W_X)_{ee} > 0$ , for all  $e \in \mathcal{E}_X$ . Given  $g \in \mathbb{R}^{n_X}$  there exists a  $\psi \in \mathbb{R}^{m_X}$  such that*

$$(2.6) \quad (\text{Div}_X \psi, q) = (g, q), \quad \text{for all } q \in \mathbb{R}^{n_X}, \quad \text{such that } (q, \mathbb{1}) = 0.$$

*Proof.* First we show that  $\mathcal{L}_X$  defined as

$$(\mathcal{L}_X p, q) = (W_X \text{Grad}_X p, \text{Grad}_X q),$$

is invertible on the sub-space of vectors orthogonal to  $\mathbb{1}$ . Indeed, assume that there exists a  $q \in \mathbb{R}^{n_X}$  with  $(q, \mathbb{1}) = 0$ , and such that  $\mathcal{L}_X q = 0$ . We then have that

$$0 = (\mathcal{L}_X q, q) = \sum_{e \in \mathcal{E}_X} (W_X)_{ee} (q_i - q_j)^2.$$

Since  $(W_X)_{ee} > 0$ , this implies that  $q_i = q_j$  for all  $(i, j) \in \mathcal{E}_X$ . By assumption, the graph  $X$  is connected, and thus, for any  $k = 1, \dots, n$ , there is a path from the vertex labeled with  $k$  to the vertex labeled with 1; namely, we have that  $q_k = q_1$  for all  $k = 1, \dots, n_X$ . This can happen if and only if  $q$  is proportional to  $\mathbb{1}$ . Since  $q$  is also  $\ell_2$ -orthogonal to  $\mathbb{1}$ , we conclude that  $q = 0$ . This implies that  $\mathcal{L}_X$  is invertible on the  $\ell_2$ -orthogonal complement of  $\text{span}\{\mathbb{1}\}$ . We denote this generalized inverse of  $\mathcal{L}_X$  with  $\mathcal{L}_X^\dagger$ . We have that  $\mathcal{L}_X \mathcal{L}_X^\dagger (g - c_0(g)\mathbb{1}) = g - c_0(g)\mathbb{1}$ , where  $c_0(g) = \langle g \rangle_X$  is the  $\ell_2(X)$ -average of  $g$ .

Letting  $\psi = -W_X \text{Grad}_X \mathcal{L}_X^\dagger (g - c_0(g)\mathbb{1})$ , we arrive at

$$\begin{aligned} (\text{Div}_X \psi, q) &= -(\psi, \text{Grad}_X(q)) = (W_X \text{Grad}_X \mathcal{L}_X^\dagger (g - c_0(g)\mathbb{1}), \text{Grad}_X q) \\ &= (\mathcal{L}_X \mathcal{L}_X^\dagger (g - c_0(g)\mathbb{1}), q) = (g - c_0(g)\mathbb{1}, q) \\ &= (g - c_0(g)\mathbb{1}, q) = (g, q). \end{aligned}$$

In the last identity, we used that  $q$  is  $\ell_2$ -orthogonal to  $\mathbb{1}$ . □

We prove a corollary from Proposition 2.1 for an aggregate  $\mathbf{a}$ , since the result stated in the corollary is used later.

**Corollary 2.2.** *Let  $(\mathcal{V}_\mathbf{a}, \mathcal{E}_\mathbf{a}) = \mathbf{a} \subset G$  be a given connected subgraph (aggregate) of a graph  $G$ . Then the problem: Find  $\psi_\mathbf{a} \in \mathbf{U}_\mathbf{a}$  satisfying*

$$(2.7) \quad (\text{Div}_G \psi_\mathbf{a}, q) = -(g, q) \quad \text{for all } q \in S_\mathbf{a}, \text{ such that } (q, \mathbb{1}) = 0,$$

*has at least one solution.*

*Proof.* We expand the left side of (2.7) to obtain that

$$(2.8) \quad (\text{Div}_G \psi_\mathbf{a}, q) = -(\psi_\mathbf{a}, \text{Grad}_G q) = - \sum_{(i,j) \in \mathcal{E}} (\psi_\mathbf{a})_e \epsilon_{ij} (q_i - q_j).$$

Since  $\psi_\mathbf{a} \in \mathbf{U}_\mathbf{a}$ , it follows that  $\psi_\mathbf{a}$  is zero on all edges not in  $\mathcal{E}_\mathbf{a}$ . Therefore, the above sum is not over all edges, but only over  $\mathcal{E}_\mathbf{a}$ , and we get

$$(2.9) \quad (\text{Div}_G \psi_\mathbf{a}, q) = - \sum_{(i,j) \in \mathcal{E}_\mathbf{a}} (\psi_\mathbf{a})_e \epsilon_{ij} (q_i - q_j) = (\text{Div}_\mathbf{a} \psi_\mathbf{a}, q).$$

We now use the fact that  $q \in S_\mathbf{a}$  to obtain that the right side of (2.7) and the orthogonality constraint  $(q, \mathbb{1}) = 0$  are as follows:

$$(2.10) \quad (g, q) = \sum_{j \in \mathcal{V}_\mathbf{a}} g_j q_j, \quad (q, \mathbb{1}) = 0 \Leftrightarrow \sum_{j \in \mathcal{V}_\mathbf{a}} q_j = 0.$$

Similar manipulations in (2.6) show that the left side, the right side and the orthogonality constraint in (2.6) are exactly the same as equations given in (2.8) and in (2.10) with  $X = \mathbf{a}$ . Thus, from the Proposition 2.1 it immediately follows that there exists at least one  $\psi_\mathbf{a} \in \mathbf{U}_\mathbf{a}$  solving (2.7). □

3. THE LOCAL BASIS OF THE COARSE EDGE SPACE  $\mathbf{U}_H$ 

We consider a collection of aggregates  $\{\mathbf{a}_k\}_{k=1}^{n_c}$  that correspond to a non-overlapping partition of the set of vertices

$$\mathcal{V} = \bigcup_{k=1}^{n_c} \mathcal{V}_{\mathbf{a}_k} = \bigcup_{\mathbf{a}} \mathcal{V}_{\mathbf{a}}.$$

Let  $e = (i, j) \in \mathcal{E}$  be a fixed edge. We then have  $i \in \mathcal{V}_{\mathbf{a}}$  and  $j \in \mathcal{V}_{\mathbf{a}'}$  with the following two possibilities: Either  $\mathbf{a} = \mathbf{a}'$  or  $\mathbf{a} \neq \mathbf{a}'$ . In the former case we have that  $e \in \mathcal{E}_{\mathbf{a}}$  and in the latter case we have that  $e$  connects two vertices from the boundaries  $\partial\mathbf{a}$  and  $\partial\mathbf{a}'$ . This splits the set of edges in two classes which we designate as follows: (1) *interior edges* denoted by  $\mathcal{E}_0$ , for which both ends are in  $\mathbf{a}$  for some  $\mathbf{a}$ ; and (2) *boundary edges*, denoted by  $\mathcal{E}_{\partial}$ , when an edge is boundary edge for exactly two aggregates, namely

$$\mathcal{E}_{\partial} = \bigcup_{\mathbf{a}, \mathbf{a}'} \mathcal{I}_{\mathbf{a}\mathbf{a}'}, \text{ with } \mathcal{I}_{\mathbf{a}\mathbf{a}'} = \{e = (i, j), e \in \mathcal{E}, \text{ such that } i \in \mathcal{V}_{\partial\mathbf{a}} \text{ and } j \in \mathcal{V}_{\partial\mathbf{a}'}\}.$$

The collections of all interfaces  $\{\mathcal{I}_{\mathbf{a}\mathbf{a}'}\}$  we denote with  $\Gamma$  and the interface edges, forming  $\Gamma$ , are exactly the edges that are in  $\mathcal{E}$ , but are not in  $\bigcup_{\mathbf{a}} \mathcal{E}_{\mathbf{a}}$ , i.e.

$$\Gamma = \mathcal{E} \setminus \left( \bigcup_{k=1}^{n_c} \mathcal{E}_{\mathbf{a}_k} \right).$$

Clearly, an interface  $\mathcal{I} = \mathcal{I}_{\mathbf{a}\mathbf{a}'} \in \Gamma$ , uniquely determines the pair  $(\mathbf{a}, \mathbf{a}')$ . In our considerations  $\mathcal{I}_{\mathbf{a}\mathbf{a}'}$  and  $\mathcal{I}_{\mathbf{a}'\mathbf{a}}$  represent one and the same interface. Thus, we need to chose only one of the  $\mathcal{I}_{\mathbf{a}\mathbf{a}'}$  or  $\mathcal{I}_{\mathbf{a}'\mathbf{a}}$  to be in  $\Gamma$ . This is the same as to choose unique labeling of the interfaces. One natural way to do this for an interface  $\mathcal{I}$  connecting a pair of aggregates  $\mathbf{a}_k$  and  $\mathbf{a}_m$  for some  $1 \leq k \leq n_c$  and  $1 \leq m \leq n_c$ ,  $k \neq m$  is to set  $\mathcal{I}_{\mathbf{a}\mathbf{a}'} \in \Gamma$  if and only if  $\mathbf{a} = \mathbf{a}_{\min\{k, m\}}$  and  $\mathbf{a}' = \mathbf{a}_{\max\{k, m\}}$ . We note here that the consideration that follow do not depend on the particular choice of labeling of the interfaces and without loss of generality we may assume that such labeling is fixed. We then introduce  $\boldsymbol{\sigma}_{\mathcal{I}}$ , for  $\mathcal{I} = \mathcal{I}_{\mathbf{a}\mathbf{a}'}$

$$(3.1) \quad \boldsymbol{\sigma}_{\mathcal{I}} = \epsilon_{\mathbf{a}, \mathbf{a}'} \mathbf{Q}_{\mathcal{I}} \text{Grad } \mathbb{1}_{\mathbf{a}}.$$

Here, the sign  $\epsilon_{\mathbf{a}, \mathbf{a}'} = \pm 1$  is chosen a priori such that  $\epsilon_{\mathbf{a}, \mathbf{a}'} = -\epsilon_{\mathbf{a}', \mathbf{a}}$ . Since the components of  $(\mathbb{1}_{\mathbf{a}} + \mathbb{1}_{\mathbf{a}'} )_k = 1$  for  $k \in \mathcal{V}_{\mathbf{a}} \cup \mathcal{V}_{\mathbf{a}'}$ , it follows that

$$\mathbf{Q}_{\mathcal{I}} \text{Grad}(\mathbb{1}_{\mathbf{a}} + \mathbb{1}_{\mathbf{a}'}) = 0.$$

This relation implies that changing  $\mathbf{a}$  to  $\mathbf{a}'$  in (3.1) gives the same quantity. Indeed, we have

$$(3.2) \quad 0 = \mathbf{Q}_{\mathcal{I}} \text{Grad}(\mathbb{1}_{\mathbf{a}} + \mathbb{1}_{\mathbf{a}'}) \implies \boldsymbol{\sigma}_{\mathcal{I}} = -\epsilon_{\mathbf{a}, \mathbf{a}'} \mathbf{Q}_{\mathcal{I}} \text{Grad } \mathbb{1}_{\mathbf{a}'} = \epsilon_{\mathbf{a}', \mathbf{a}} \mathbf{Q}_{\mathcal{I}} \text{Grad } \mathbb{1}_{\mathbf{a}'}.$$

For a given  $\boldsymbol{\psi} \in \mathbf{U}$  we now introduce the following averaging operator, which also respects the directions of the edges on the interface  $\mathcal{I}_{\mathbf{a}\mathbf{a}'}$ :

$$(3.3) \quad \{\!\!\{\boldsymbol{\psi}\}\!\!\}_{\mathcal{I}} = \frac{(\boldsymbol{\psi}, \boldsymbol{\sigma}_{\mathcal{I}})}{\|\boldsymbol{\sigma}_{\mathcal{I}}\|^2}.$$

**Example 3.1.** *As an example, let us consider the case when all edges on the interface are pointing in one direction, or equivalently,  $\epsilon_{ij} = 1$ , for all  $(i, j) = e \in \mathcal{I}$  with  $i \in \partial\mathcal{V}_{\mathbf{a}}$  and  $j \in \partial\mathcal{V}_{\mathbf{a}'}$ . Letting  $\epsilon_{\mathbf{a}, \mathbf{a}'} = 1$ , we have  $\boldsymbol{\sigma}_{\mathcal{I}} = \mathbb{1}_{\mathcal{I}}$  and  $\{\!\!\{\boldsymbol{\psi}\}\!\!\}_{\mathcal{I}} = \langle \boldsymbol{\psi} \rangle_{\mathcal{I}}$ .*

To construct a coarse basis and associated projection  $\pi_H$ , we need a bilinear form

$$(3.4) \quad \mathcal{B}(\cdot, \cdot) : \mathbf{U} \times \mathbf{U} \mapsto \mathbb{R},$$

that is symmetric and positive semi-definite on  $\mathbf{U}$ . We require that its restriction to the local spaces  $\mathbf{U}_{\mathcal{E}_{\mathbf{a}}}$  be positive definite.

We are interested in the following local saddle-point problem: Find  $\varphi_{\mathcal{I},\mathbf{a}} \in \mathbf{U}_{\mathcal{E}_{\mathbf{a}}}$  and  $p_{\mathcal{I},\mathbf{a}} \in S_{\mathbf{a}}$ ,  $(p, \mathbb{1}) = 0$ , satisfying

$$(3.5) \quad \mathcal{B}(\varphi_{\mathcal{I},\mathbf{a}}, \mathbf{v}) + (p_{\mathcal{I},\mathbf{a}}, \text{Div } \mathbf{v}) = -\mathcal{B}(\sigma_{\mathcal{I}}, \mathbf{v}), \quad \text{for all } \mathbf{v} \in \mathbf{U}_{\mathcal{E}_{\mathbf{a}}}$$

$$(3.6) \quad (\text{Div } \varphi_{\mathcal{I},\mathbf{a}}, q) = -(\text{Div } \sigma_{\mathcal{I}}, q),$$

for all  $q \in S_{\mathbf{a}}$ , such that  $(q, \mathbb{1}) = 0$ .

An example of (3.4) is the  $\ell_2$ -inner product on  $\mathbf{U}$ , namely:

$$(3.7) \quad \mathcal{B}(\psi, \eta) = \sum_{e \in \mathcal{E}} \psi_e \eta_e.$$

Note that this choice, the right hand side in (3.5) is zero (since the support  $\mathcal{I}$  of  $\sigma_{\mathcal{I}}$  and the support of  $\mathbf{v} \in \mathbf{U}_{\mathcal{E}_{\mathbf{a}}}$  are non-intersecting). Another example of (3.4) is given in Example 4.1.

The saddle point problem (3.5)-(3.6) clearly has a solution. Indeed, we observe that this is the Lagrange multiplier formulation of the constrained minimization problem:

$$(3.8) \quad \varphi_{\mathcal{I},\mathbf{a}} = \arg \min \{ \mathcal{B}(\psi + \sigma_{\mathcal{I}}, \psi + \sigma_{\mathcal{I}}) \}, \quad \text{subject to } \psi \in \mathbf{U}_{\mathcal{E}_{\mathbf{a}}}, \quad \text{and}$$

$$(3.9) \quad (\text{Div } \psi, q) = -(\text{Div } \sigma_{\mathcal{I}}, q), \quad \text{for all } q \in S_{\mathbf{a}}, \quad \text{such that } (q, \mathbb{1}) = 0.$$

By Corollary 2.2, the set of functions that satisfy the constraints is non-empty. It follows then that the minimizer  $\varphi_{\mathcal{I},\mathbf{a}}$  exists and is unique, because  $\mathcal{B}(\cdot, \cdot)$  is SPD on  $\mathbf{U}_{\mathcal{E}_{\mathbf{a}}}$ , as follows from the classical theory of saddle point problems. We solve analogous problem on  $\mathbf{a}'$  and we find a solution  $\varphi_{\mathcal{I},\mathbf{a}'}$  there. We then define  $\varphi_{\mathcal{I}}$  (which corresponds to  $\mathcal{I}$ ) in the following way:

$$(\varphi_{\mathcal{I}})_e = \begin{cases} (\varphi_{\mathcal{I},\mathbf{a}})_e & \text{for all } e \in \mathcal{E}_{\mathbf{a}}, \\ (\sigma_{\mathcal{I}})_e & \text{for all } e \in \mathcal{I}, \\ (\varphi_{\mathcal{I},\mathbf{a}'} )_e & \text{for all } e \in \mathcal{E}_{\mathbf{a}'}, \\ 0 & \text{for all other edges.} \end{cases}$$

Note that from the second equation in (3.5)-(3.6) we get that

$$(3.10) \quad (\text{Div } \varphi_{\mathcal{I}}, q) = 0, \quad \text{for all } q \in S_{\mathbf{a}}, \quad \text{such that } (q, \mathbb{1}) = 0.$$

We now define the piece-wise constant coarse space

$$S_H = \oplus_{k=1}^{n_c} \text{span}\{Q_{\mathbf{a}_k} \mathbb{1}\} = \oplus_{k=1}^{n_c} \text{span}\{\mathbb{1}_{\mathbf{a}_k}\}.$$

Since the sets of vertices  $\mathcal{V}_{\mathbf{a}_k}$ ,  $k = 1, \dots, n_c$  are disjoint the  $\ell_2$ -orthogonal projection on the space  $S_H$  is given by

$$Q_H = \sum_{k=1}^{n_c} Q_{0,\mathbf{a}_k} = \sum_{\mathbf{a}} Q_{0,\mathbf{a}}.$$

Finally, we define the coarse subspace of  $\mathbf{U}$  as

$$(3.11) \quad \mathbf{U}_H = \text{span}\{\varphi_{\mathcal{I}}\}_{\mathcal{I} \in \Gamma}.$$

#### 4. EDGE SPACES AND A GLOBAL ENERGY MINIMIZING PROPERTY OF THE BASIS $\{\varphi_I\}$

We now focus on a global property of the basis we have defined for  $\mathbf{U}_H$ , namely  $\{\varphi_I\}_{I \in \Gamma}$ . These functions are defined via the interface conditions  $(\varphi_I)_e = \sigma_e$  for all  $e \in \mathcal{I}$ . Our goal is to show a global energy minimization property of their sum. This can be done for a wide range of bilinear forms  $\mathcal{B}(\cdot, \cdot)$  in (3.4). More specifically, we assume that the form  $\mathcal{B}$  gives rise to a matrix  $B = (b_{e,e'})$  with graph corresponding to its sparsity for which the interface  $\Gamma$  is a separator, i.e.,  $B$  has possibly nonzero entries  $b_{e,e'}$  if  $e \in \mathcal{E}_a$  only for  $e' \in \mathcal{E}_a \cup \Gamma$  but not for  $e' \in \mathcal{E}_{a'}$ , for any  $a' \neq a$ . Therefore, we have the localization property

$$(4.1) \quad \mathcal{B}(\mathbf{u}, \mathbf{v}) = 0, \text{ for any } \mathbf{u} \in \mathbf{U}_{\mathcal{E}_a} \text{ and any } \mathbf{v} \in \mathbf{U}_{\mathcal{E}_{a'}}, a \neq a'.$$

It is clear that the  $\ell_2$ -form (3.7) trivially satisfies the above localization property.

**Example 4.1.** Another example of such global bilinear form  $\mathcal{B}(\cdot, \cdot)$  satisfying (4.1) is defined via a graph, which is dual, in some sense, to  $G = (\mathcal{V}, \mathcal{E})$ . More specifically, we consider  $G^* = (\mathcal{V}^*, \mathcal{E}^*)$ , where  $\mathcal{V}^* = \mathcal{E}$ , and a pair  $(e, e') \in \mathcal{E}^*$ , for  $e \in \mathcal{E}$  and  $e' \in \mathcal{E}$ , if and only if  $e$  and  $e'$  share a vertex. Next, consider the graph Laplacian on  $G^*$ , denoted with  $\mathcal{L}_{G^*} : \mathbf{U} \mapsto \mathbf{U}$ . For the bilinear form  $\mathcal{B}$ , we then put:

$$(4.2) \quad \mathcal{B}(\boldsymbol{\varphi}, \boldsymbol{\psi}) = (\mathcal{L}_{G^*}(\boldsymbol{\epsilon}\boldsymbol{\varphi}), \boldsymbol{\epsilon}\boldsymbol{\psi}).$$

Here,  $\boldsymbol{\epsilon}\boldsymbol{\eta}$  is an edge-defined function (i.e., from  $\mathbf{U}$ ) such that  $(\boldsymbol{\epsilon}\boldsymbol{\eta})_e = \epsilon_e \eta_e$ . The entries  $\epsilon_e = \epsilon_{ij}$ ,  $e = (i, j)$  are the ones used to define Grad-operator in (2.1).

Consider the edge-sign function  $\boldsymbol{\epsilon}$  introduced in Example 4.1. For any vertex  $i \in V_G$  and  $\boldsymbol{\delta}_i = (\delta_{i,j})_{j=1}^n$  the unit coordinate vector associated with it, for the  $i$ th entry of  $\text{Div } \boldsymbol{\epsilon}$ , we have

$$\begin{aligned} (\text{Div } \boldsymbol{\epsilon})_i &= -(\boldsymbol{\epsilon}, \text{Grad } \boldsymbol{\delta}_i) = - \sum_{e=(i,j) \in \mathcal{E}} \epsilon_e (\text{Grad } \boldsymbol{\delta}_i)_e \\ &= - \sum_{e=(i,j) \in \mathcal{E}} \epsilon_e (\epsilon_e (\delta_{ii} - \delta_{ij})) = - \sum_{e=(i,j) \in \mathcal{E}} 1 \\ &= -\deg(i). \end{aligned}$$

Above,  $\deg(i)$  stands for the number of edges meeting at vertex  $i$ , i.e., it denotes the degree of vertex  $i$ .

It is clear then that with  $D = \text{diag}(\deg(i))$ ,

$$(4.3) \quad \text{Div } \boldsymbol{\epsilon} = -D\mathbf{1}.$$

For bilinear forms  $\mathcal{B}$  that satisfy the localization property (4.1), we have the following main characterization result of the basis  $\{\varphi_I\}_{I \in \Gamma}$ .

**Theorem 4.2.** Let  $\boldsymbol{\varphi} = \sum_{I \in \Gamma} \varphi_I$ , where  $\varphi_I$  is defined based on the boundary value  $\sigma_I$  and the functions  $\varphi_a$  and  $\varphi_{a'}$  which are the solution of the local constrained minimization problem (3.8) (for  $a$  and  $a'$ , respectively). If the energy form  $\mathcal{B}$  satisfies the localization property (4.1), then  $\boldsymbol{\varphi} = \sum_{I \in \Gamma} \varphi_I$  is the unique minimizer of the global constrained minimization problem

$$(4.4) \quad \mathcal{B}(\boldsymbol{\varphi}, \boldsymbol{\varphi}) \mapsto \min,$$

such that

$$(4.5) \quad \varphi|_{\Gamma} = \sigma_{\Gamma} \equiv \sum_{\mathcal{I} \in \Gamma} \sigma_{\mathcal{I}},$$

$$(\text{Div } \varphi, q) = 0, \text{ for all } q \in S : (q, \mathbb{1}_{\mathbf{a}}) = 0, \text{ for all } \mathbf{a}.$$

*Proof.* By construction, we have that each basis function  $\varphi_{\mathcal{I}} = \varphi_{\mathcal{I}, \mathbf{a}} + \varphi_{\mathcal{I}, \mathbf{a}'} + \sigma_{\mathcal{I}}$  and  $p_{\mathcal{I}} \in S_{\mathbf{a}} \oplus S_{\mathbf{a}'}$  being equal to  $p_{\mathcal{I}, \mathbf{a}}$  on  $\mathbf{a}$  and to  $p_{\mathcal{I}, \mathbf{a}'}$  on  $\mathbf{a}'$ , satisfy:

$$(4.6) \quad \mathcal{B}(\varphi_{\mathcal{I}}, \mathbf{v}) + (p_{\mathcal{I}}, \text{Div } \mathbf{v}) = 0, \quad \text{for all } \mathbf{v} \in \mathbf{U}_{\mathcal{E}_{\mathbf{a}}} \oplus \mathbf{U}_{\mathcal{E}_{\mathbf{a}'}},$$

$$(4.7) \quad (\text{Div } \varphi_{\mathcal{I}}, q) = 0,$$

$$(4.8) \quad \begin{aligned} & \text{for all } q \in S_{\mathbf{a}} \oplus S_{\mathbf{a}'}, \\ & \text{such that } (q, \mathbb{1}_{\mathbf{a}}) = 0 \text{ and } (q, \mathbb{1}_{\mathbf{a}'}) = 0. \end{aligned}$$

Note that equation (4.6) above holds true also for  $\mathbf{v} \in \oplus_{\mathbf{a}'} \mathbf{U}_{\mathcal{E}_{\mathbf{a}'}}$ , i.e., for any  $\mathbf{v} \in \mathbf{U}$  vanishing on  $\Gamma$  due to the localization property (4.1) and the local support of  $p_{\mathcal{I}}$ . Similarly, equation (4.7) holds for any  $q \in \oplus_{\mathbf{a}'} S_{\mathbf{a}'}$ .

$$(4.9) \quad \mathcal{B}(\varphi, \mathbf{v}) + \left( \sum_{\mathcal{I}} p_{\mathcal{I}}, \text{Div } \mathbf{v} \right) = 0, \quad \text{for all } \mathbf{v} \in \oplus_{\mathbf{a}} \mathbf{U}_{\mathcal{E}_{\mathbf{a}}}$$

$$(4.10) \quad (\text{Div } \varphi, q) = 0, \quad \text{for all } q \in \oplus_A S_{\mathbf{a}}, \text{ such that } (q, \mathbb{1}_{\mathbf{a}}) = 0.$$

Observing now that the equations above are the Lagrange equations for the global constraint minimization problem (4.4)-(4.5), we conclude that  $\varphi$  in fact solves that minimization problem.  $\square$

We now apply Theorem (4.2) to a graph with constant degree vertices. In some applications, it is useful to embed a given graph Laplacian into a graph Laplacian corresponding to a larger graph, however with constant degree. In that case, we have the following important property of the basis  $\{\varphi_{\mathcal{I}}\}$ , since it provides partition of unity (or rather partition of the edge-sign function  $\epsilon$ ) for the edge spaces. If we take the bilinear form defined via the dual graph Laplacian in (4.2), when the signs  $\varepsilon_{ij}$ ,  $(i, j) = e \in \mathcal{E}$  are chosen so that  $\sigma_{\mathcal{I}} = \mathbb{1}_{\mathcal{I}}$  the basis  $\varphi_{\mathcal{I}}$  forms partition of unity, namely,  $\sum_{\mathcal{I} \in \Gamma} \varphi_{\mathcal{I}} = \epsilon$ . This identity follows from the fact

that  $\sum_{\mathcal{I} \in \Gamma} \varphi_{\mathcal{I}}$  minimizes  $\mathcal{B}(\cdot, \cdot)$  and the unique minimizer in this case is  $\epsilon \in \mathbf{U}$ . Note that

$\sum_{\mathcal{I} \in \Gamma} \varphi_{\mathcal{I}}$  restricted to  $\mathcal{I}$  equals  $\varphi_{\mathcal{I}} = \sigma_{\mathcal{I}} = \epsilon_{\mathbf{a}, \mathbf{a}'} \text{Grad } \mathbb{1}_{\mathbf{a}} = \epsilon_{\mathbf{a}, \mathbf{a}'} \mathbf{Q}_{\mathcal{I}} \epsilon$  which equals  $\epsilon$  restricted

to  $\mathcal{I}$ . Also, due to (4.3), we have  $(\text{Div } \epsilon, \mathbf{Q}_{\mathbf{a}} q) = (D \mathbb{1}, \mathbf{Q}_{\mathbf{a}} q) = \text{const } (q, \mathbb{1}_A) = 0$ , i.e., it satisfies the constraints (4.5) since for the graph we consider,  $D = \text{diag } (\deg(i)) = \text{const } I$ .

## 5. COMMUTING DIAGRAM PROPERTY

We aim to show now that with properly constructed  $\pi_H : \mathbf{U} \mapsto \mathbf{U}_H$  the following diagram

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{\text{Div}} & S \\ \pi_H \downarrow & & \downarrow Q_H \\ \mathbf{U}_H & \xrightarrow{\text{Div}} & S_H \end{array}$$

commutes. Above,  $Q_H$  is the  $\ell_2$ -based projection on the space  $S_H$ . We need the following proposition, which shows an important property for the locally constructed  $\varphi_{\mathcal{I}}$ .

**Proposition 5.1.** *Let  $\mathcal{I} \in \Gamma$  be such that it connects two aggregates  $\mathbf{a}$  and  $\mathbf{a}'$ , and  $q \in \mathbb{R}^n$ . We then have the following*

$$(\text{Div } \varphi_{\mathcal{I}}, q) = (Q_H \text{Div } \varphi_{\mathcal{I}}, q).$$

*Proof.* Let  $\mathbf{a}$  be any of the aggregates adjacent to  $\mathcal{I}$ . By construction  $(\text{Div } \varphi_{\mathcal{I}}, Q_{\mathbf{a}}q) = 0$ , for all  $q$  such that  $(Q_{\mathbf{a}}q, \mathbb{1}) = 0$  (see equation (3.10)). Observing that  $\sum_{\mathbf{a}} Q_{\mathbf{a}}q = q$ , and

$$(5.1) \quad Q_{0,\mathbf{a}} = Q_{\mathbf{a}}Q_{0,\mathbf{a}} = Q_{0,\mathbf{a}}Q_{\mathbf{a}},$$

we obtain

$$\begin{aligned} (\text{Div } \varphi_{\mathcal{I}}, q) &= (\text{Div } \varphi_{\mathcal{I}}, \sum_{\mathbf{a}} Q_{\mathbf{a}}q) = \sum_{\mathbf{a}} (\text{Div } \varphi_{\mathcal{I}}, Q_{\mathbf{a}}q) \\ &= \sum_{\mathbf{a}} (\text{Div } \varphi_{\mathcal{I}}, Q_{0,\mathbf{a}}Q_{\mathbf{a}}q) + \sum_{\mathbf{a}} (\text{Div } \varphi_{\mathcal{I}}, (I - Q_{0,\mathbf{a}})Q_{\mathbf{a}}q) \\ &= \sum_{\mathbf{a}} (\text{Div } \varphi_{\mathcal{I}}, Q_{0,\mathbf{a}}q) = \sum_{\mathbf{a}} (Q_{0,\mathbf{a}} \text{Div } \varphi_{\mathcal{I}}, q) \\ &= \left( \sum_{\mathbf{a}} Q_{0,\mathbf{a}} \text{Div } \varphi_{\mathcal{I}}, q \right) = (Q_H \text{Div } \varphi_{\mathcal{I}}, q). \end{aligned}$$

The first identity holds because the aggregates are disjoint, the second follows the fact that the scalar product is bi-linear, the third is just splitting  $Q_{\mathbf{a}}q$  as a sum of constant part and a part orthogonal to the constant on  $\mathbf{a}$ , namely,

$$Q_{\mathbf{a}}q = Q_{0,\mathbf{a}}Q_{\mathbf{a}}q + (I - Q_{0,\mathbf{a}})Q_{\mathbf{a}}q.$$

The fourth equality holds by (3.10) and the rest of the relations follow easily from the definitions and equation (5.1).  $\square$

The definition of interpolation from the space  $\text{span}\{\varphi_{\mathcal{I}}\}_{\mathcal{I} \in \Gamma}$  is done then in a canonical way:

**Definition 5.2.** *For any  $\psi \in U$ , we set*

$$(5.2) \quad \pi_H \psi = \sum_{\mathcal{I} \in \Gamma} \{\{\psi\}\}_{\mathcal{I}} \varphi_{\mathcal{I}}.$$

By construction  $\pi_H : U \mapsto U_H$ . We next prove a theorem showing that the diagram given at the beginning of this section commutes.

**Theorem 5.3.** *For all  $\psi \in U$  and  $q \in S$  we have*

$$(\text{Div } \pi_H \psi, q) = (Q_H \text{Div } \psi, q).$$

*Proof.* From Proposition 5.1 and the definition of Grad we have

$$\begin{aligned}
-(\text{Div } \pi_H \psi, q) &= - \sum_{\mathcal{I} \in \Gamma} \{\psi\}_{\mathcal{I}} (\text{Div } \varphi_{\mathcal{I}}, q) \\
&= - \sum_{\mathcal{I} \in \Gamma} \{\psi\}_{\mathcal{I}} (\text{Div } \varphi_{\mathcal{I}}, Q_H q) \\
&= \sum_{\mathcal{I} \in \Gamma} \{\psi\}_{\mathcal{I}} (\varphi_{\mathcal{I}}, \text{Grad}(Q_H q)).
\end{aligned}$$

Above, we have used that  $\text{Div } \varphi_{\mathcal{I}}$  is constant over each aggregate  $\mathbf{a}$ . Next, observe that  $(\text{Grad } Q_H q)_e = 0$ , for all  $e \notin \Gamma$ , and hence,  $Q_{\mathcal{I}}(\text{Grad } Q_H q) = (\text{Grad } Q_H q)$ . We then have, for all  $\eta \in \mathcal{U}$  and  $q \in S$ ,

$$(5.3) \quad (\eta, \text{Grad } Q_H q) = \sum_{\mathcal{I} \in \Gamma} (\eta, Q_{\mathcal{I}}(\text{Grad } Q_H q)) = \sum_{\mathcal{I} \in \Gamma} (Q_{\mathcal{I}} \eta, Q_{\mathcal{I}}(\text{Grad } Q_H q))$$

From equations (3.1) and (3.2) we also obtain

$$\begin{aligned}
(5.4) \quad Q_{\mathcal{I}}(\text{Grad } Q_H q) &= Q_{\mathcal{I}} \text{Grad}(\langle q \rangle_{\mathbf{a}} \mathbb{1}_{\mathbf{a}} + \langle q \rangle_{\mathbf{a}'} \mathbb{1}_{\mathbf{a}'}) \\
&= \langle q \rangle_{\mathbf{a}} Q_{\mathcal{I}} \text{Grad } \mathbb{1}_{\mathbf{a}} + \langle q \rangle_{\mathbf{a}'} Q_{\mathcal{I}} \text{Grad } \mathbb{1}_{\mathbf{a}'} \\
&= \epsilon_{\mathbf{a}, \mathbf{a}'} (\langle q \rangle_{\mathbf{a}} - \langle q \rangle_{\mathbf{a}'}) \sigma_{\mathcal{I}}.
\end{aligned}$$

Using now the relations (5.3) and (5.4) (both are used twice), the fact that  $Q_{\mathcal{I}} \varphi_{\mathcal{I}} = \sigma_{\mathcal{I}}$ , and the defining relation for  $\{\psi\}_{\mathcal{I}}$  in (3.3), we arrive at

$$\begin{aligned}
-(\text{Div } \pi_H \psi, q) &= \sum_{\mathcal{I} \in \Gamma} \{\psi\}_{\mathcal{I}} (Q_{\mathcal{I}} \varphi_{\mathcal{I}}, Q_{\mathcal{I}}(\text{Grad } Q_H q)) \\
&= \sum_{\mathcal{I} \in \Gamma} \{\psi\}_{\mathcal{I}} (\sigma_{\mathcal{I}}, Q_{\mathcal{I}}(\text{Grad } Q_H q)) = \sum_{\mathcal{I} \in \Gamma} \{\psi\}_{\mathcal{I}} \epsilon_{\mathbf{a}, \mathbf{a}'} (\langle q \rangle_{\mathbf{a}} - \langle q \rangle_{\mathbf{a}'}) \|\sigma_{\mathcal{I}}\|^2 \\
&= \sum_{\mathcal{I} \in \Gamma} (\psi, \sigma_{\mathcal{I}}) \epsilon_{\mathbf{a}, \mathbf{a}'} (\langle q \rangle_{\mathbf{a}} - \langle q \rangle_{\mathbf{a}'}) = \sum_{\mathcal{I} \in \Gamma} (\psi, \epsilon_{\mathbf{a}, \mathbf{a}'} (\langle q \rangle_{\mathbf{a}} - \langle q \rangle_{\mathbf{a}'})) \sigma_{\mathcal{I}} \\
&= \sum_{\mathcal{I} \in \Gamma} (\psi, Q_{\mathcal{I}} \text{Grad } Q_H q) = (\psi, \text{Grad } Q_H q) \\
&= -(\text{Div } \psi, Q_H q) = -(Q_H \text{Div } \psi, q).
\end{aligned}$$

This completes the proof. □

Using adjoint operations, we obtain

$$(\text{Div } \pi_H \psi, q) = -(\pi_H \psi, \text{Grad } q) = -(\psi, \pi_H^* \text{Grad } q).$$

Similarly,

$$(Q_H \text{Div } \psi, q) = (\text{Div } \psi, Q_H q) = -(\psi, \text{Grad } Q_H q).$$

Then Theorem 5.3 implies the following corollary.

**Corollary 5.4.** *Let  $\pi_H^*$  be the  $\ell_2$ -adjoint of  $\pi_H$ . Then, the following identity holds:*

$$\pi_H^* \text{Grad} = \text{Grad } Q_H.$$

## 6. NUMERICAL ILLUSTRATION

**6.1. Illustration of graphs, aggregation, and properties of the coarse edge-space  $U_H$ .** We plot here several figures illustrating the notions we have introduced earlier and the results we have shown to hold. In Figure 6.1 we show the example of a graph which we use for our test: a planar triangulation of L-shaped domain with 1016 vertices and 2926 edges. Next to it in Fig. 6.1 we show the splitting into 14 subgraphs obtained by a simple and well known aggregation algorithm. The greedy aggregation algorithm that we use is probably

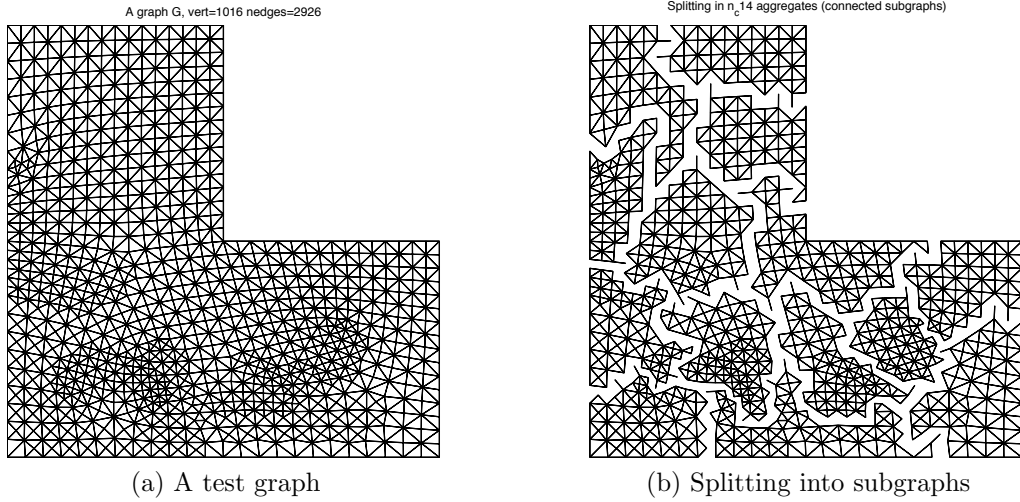


Figure 6.1: A planar graph  $G$  with 1016 vertices and 2926 edges, and a splitting of  $G$  into subgraphs

the simplest version discussed in [11] is as follows:

**Algorithm 6.1.** *Greedy subgraph splitting.*

- (1) Set  $nc = 0$  and for  $i = 1 : n$  do:
  - (a) If  $i$  and all its neighbors have not been visited, then we set  $nc = nc + 1$  and mark  $i$  and all its neighbors as visited and also being in the subgraph  $nc$ .
  - (b) If at least one neighbor of  $i$  has been visited, we continue the loop over the vertices.
- (2) Since after this procedure there will be vertices which do not belong to any aggregate (but definitely have a neighboring aggregate), we add each such vertex to a neighboring aggregate and we pick the one which has minimal number of points in it.
- (3) After such pass all vertices of the graph are members of an aggregate.

One may apply this algorithm recursively, by defining a coarse grid graph in which the vertices are the aggregates just formed and we have an edge between two aggregates  $\mathbf{a}$  and  $\mathbf{a}'$  if and only if there is an edge in the graph  $G$  connecting a vertex in  $\mathcal{V}_{\mathbf{a}}$  with a vertex in  $\mathcal{V}_{\mathbf{a}'}$ . One then can apply the same algorithms to form aggregates of aggregates, etc. The splitting depicted in Fig. 6.1 is obtained via two such recursive aggregations and its coarse grid graph is shown in Fig 6.2.

In the same Figure 6.2, we have plotted a spanning tree for each of the subgraphs (so in our example there are 14 spanning trees shown in Fig. 6.2. If  $T_{\mathbf{a}}$  is such a tree for aggregate

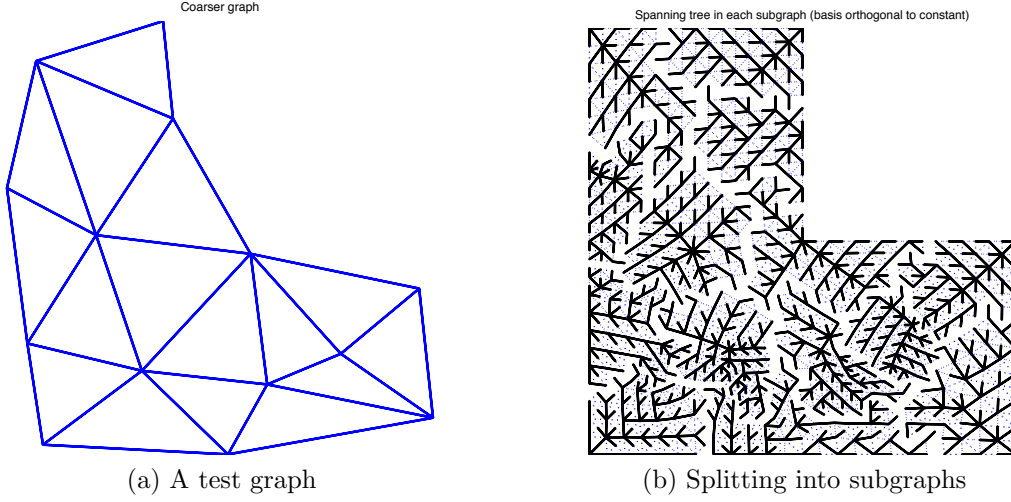


Figure 6.2: Coarse grid graph and spanning trees providing sparse basis orthogonal to the constant in each of the aggregates

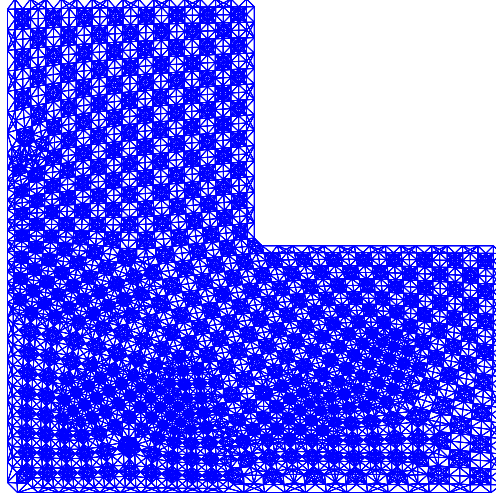


Figure 6.3: Illustration of graph  $G^*$  introduced in Example 4.1

$\mathbf{a}$ , a sparse basis orthogonal to the constant in  $\mathbf{a}$ , (namely a basis for the space orthogonal to  $\mathbb{1}_{\mathbf{a}}$ ) is provided by the columns of  $\text{Div}_{T_{\mathbf{a}}}$ . This is immediate consequence from the fact that a connected tree exists for each connected subgraph and such tree has exactly  $(|\mathcal{V}_{\mathbf{a}}| - 1)$  edges and also from the fact that when restricted to  $\mathbf{a}$  the vector  $\mathbb{1}_{\mathbf{a}}$  is a basis for the kernel of columns  $\text{Grad}_{T_{\mathbf{a}}}$ .

In Fig. 6.3 we show the graph  $G^*$  used to define the bilinear form in (4.2) and is restriction of the bilinear form given by the graph Laplacian  $\mathcal{L}_{G^*}$  in Example 4.1. To illustrate the result of Proposition 5.1 we have also shown in Fig. 6.4 the plot of  $\text{Div } \psi_H$ , where  $\psi_H$  is

$$\psi_H = \sum_{I \in \Gamma} \alpha_I \varphi_I,$$

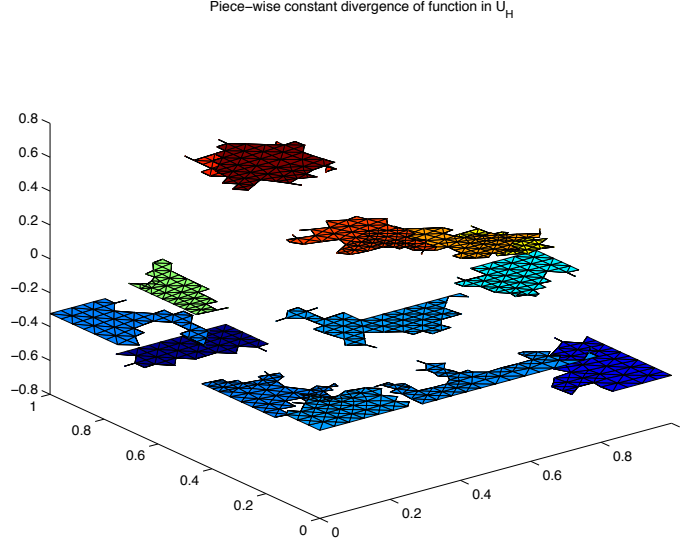


Figure 6.4: Plot of the divergence  $\text{Div } \psi_H$  for  $\psi_H \in U_H$ .

with  $\{\alpha_I\}_{I \in \Gamma}$  are chosen at random. By Proposition 5.1 this should be a piece-wise constant, as is clearly seen in Fig. 6.4.

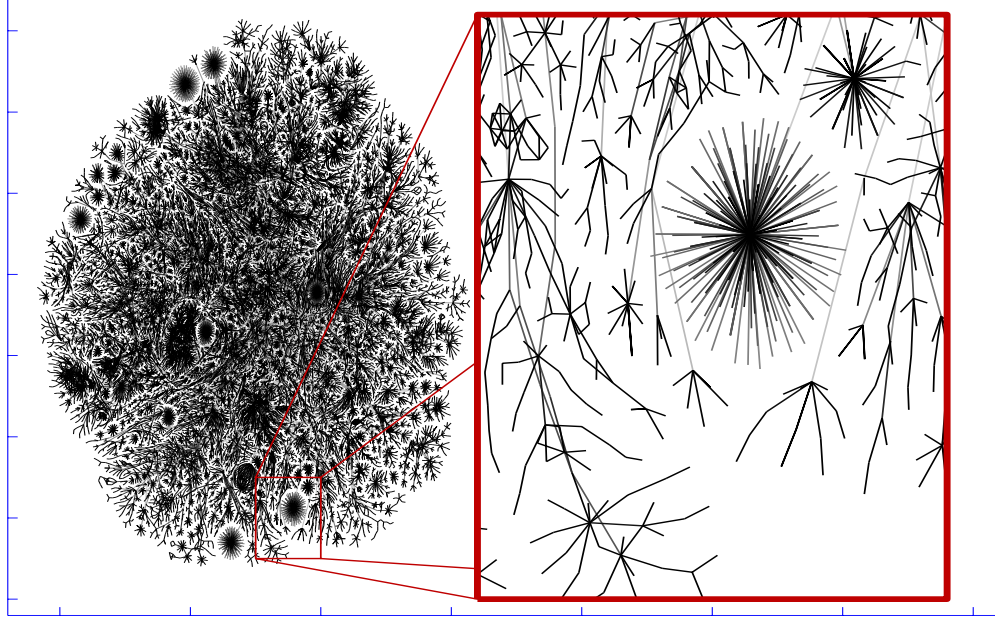


Figure 6.5: An internet graph from <http://opte.org>

Another example we downloaded from <http://opte.org> and is the graph of the connections between class C networks on the internet. In Fig. 6.5 we have plotted the graph and in Fig. 6.6 a subgraph with its neighboring subgraphs is shown. The colors indicate different values of the piece-wise constant divergence of a vector in  $U_H$ .

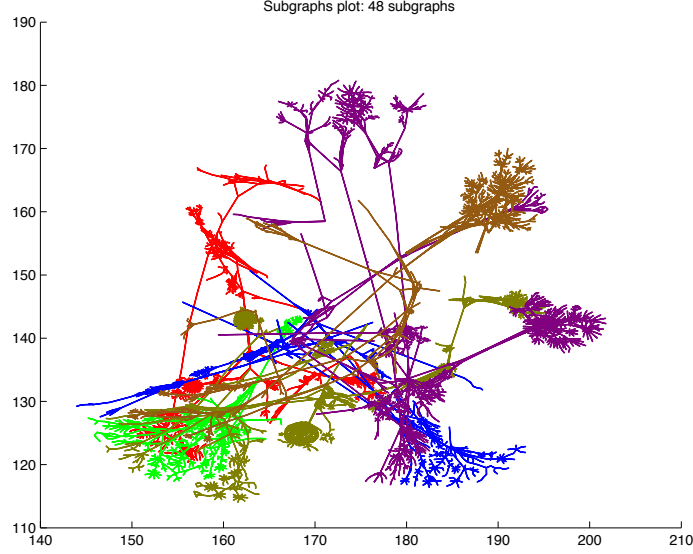


Figure 6.6: Plot of the neighborhood of a subgraph. Different colors indicate the different values of the piece-wise constant divergence.

**6.2. A two-level method.** We have also investigated the convergence of a two-level Schwarz method for the mixed formulation of the graph Laplacian, namely: Find  $\mathbf{u} \in \mathbf{U}$  and  $p \in S$  such that

$$(6.1) \quad \begin{aligned} \mathcal{B}(\mathbf{u}, \mathbf{v}) + (\text{Div } \mathbf{v}, p) &= 0, \text{ for all } \mathbf{v} \in \mathbf{U}, \\ (\text{Div } \mathbf{u}, q) &= (-f, q), \text{ for all } q \in S. \end{aligned}$$

We note that here and in what follows, we consider  $f$  orthogonal to constants.

The two-level method that we consider here is as follows. Let

$$\mathbf{U}_{\mathbf{aa}'} = \mathbf{U}_{\mathbf{a}} \oplus \mathbf{U}_{\mathbf{a}'} \oplus \mathbf{U}_{\mathcal{I}_{\mathbf{aa}'}}}, \quad S_{\mathbf{aa}'} = S_{\mathbf{a}} \oplus S_{\mathbf{a}'},$$

be the spaces associated with a pair of aggregates  $\mathbf{a}$  and  $\mathbf{a}'$  which are neighboring, namely  $\mathcal{I}_{\mathbf{aa}'} \subset \Gamma$ . Let  $\mathcal{E}_c$  be the set of such pairs (coarse graph edges). The two-level Schwarz is then as follows (here  $J = n_{\mathcal{E}_c} = |\mathcal{E}_c|$ ):

**Algorithm 6.2.** *Two-level Schwarz algorithm:*

1. Let  $\mathbf{u}_0$  be such that  $\text{Div } \mathbf{u}_0 = -f$ .
2. For  $k = 1, \dots$  until convergence
3. Set  $\mathbf{w}_0 = \mathbf{u}_{k-1}$  and for  $\ell = 1, \dots, J$  (i.e., for every pair  $\ell = (\mathbf{a}, \mathbf{a}') \in \mathcal{E}_c$ ), solve the constrained minimization problem

$$\mathbf{v}_\ell = \arg \min_{\boldsymbol{\chi}_\ell \in \mathbf{U}_{\mathbf{aa}'}} \mathcal{B}(\mathbf{w}_{\ell-1} + \boldsymbol{\chi}_\ell, \mathbf{w}_{\ell-1} + \boldsymbol{\chi}_\ell),$$

subject to  $(\text{Div } \mathbf{v}_\ell, q) = 0$ , for all  $q \in S_{\mathbf{aa}'}$ . Set  $\mathbf{w}_\ell = \mathbf{w}_{\ell-1} + \mathbf{v}_\ell$ .

4. Correct on the coarse grid by solving:

$$\mathbf{v}_H = \arg \min_{\boldsymbol{\chi}_H \in \mathbf{U}_H} \mathcal{B}(\mathbf{w}_J + \boldsymbol{\chi}_H, \mathbf{w}_J + \boldsymbol{\chi}_H),$$

subject to  $(\text{Div } \mathbf{v}_H, q) = 0$ , for all  $q \in S_H$ .

5. Finally, the next iterate is  $\mathbf{u}_k = \mathbf{w}_J + \mathbf{v}_H$ .

The initial guess  $\mathbf{u}_0$  is found by first constructing a spanning tree, say a breath-first-search tree (cf., e.g., [4]),  $T$  of  $G$ . Then, we solve for  $\tilde{\mathbf{u}}_{0,T} \in \mathbf{U}_T$  (i.e.  $\tilde{\mathbf{u}}_{0,T}$  is defined only on the tree edges):

$$(6.2) \quad \text{Div}_T \tilde{\mathbf{u}}_{0,T} = -f.$$

Let  $\mathcal{E}_T$  be the set of the tree edges. We then set  $\mathbf{u}_0|_{\mathcal{E}_T} = \tilde{\mathbf{u}}_{0,T}$  and  $\mathbf{u}_0 = 0$  on the other edges. It is easy to see that by construction,  $\text{Div}_G \mathbf{u}_0 = \text{Div}_T \tilde{\mathbf{u}}_{0,T} = -f$  (since by assumption  $f$  is orthogonal to constants). We also note that (6.2) is solvable with optimal cost, by the following proposition.

**Proposition 6.3.** *Let  $T$  be a tree with  $n$  edges and define  $\mathcal{G} = \text{Grad}_T$ . Then  $\mathcal{G}$  is invertible (on the space orthogonal to  $\mathbb{1}$ ) in  $\mathcal{O}(n)$  operations. Similarly, consider  $\text{Div}_T$  (the adjoint of  $\mathcal{G}$ ). Then the problem (6.2) is solvable for any r.h.s. orthogonal to  $\mathbb{1}$  also in linear time.*

*Proof.* Since  $T$  is a tree it connects  $p \geq 1$  subtrees  $T_1, \dots, T_p$  with a parent vertex. Let  $\mathcal{G}_i$  be the restriction of  $\text{Grad}_T$  to the subtree  $T_i$ . Then the matrix representation of  $\mathcal{G} = \text{Grad}_T$  admits the following (upper) triangular form:

$$\mathcal{G} = \begin{bmatrix} \epsilon_1 & [0, -\epsilon_1, 0] & 0 & \dots & 0 \\ \epsilon_2 & 0 & [0, -\epsilon_2, 0] & \dots & 0 \\ \vdots & & & & 0 \\ \epsilon_p & 0 & \dots & \dots & [0, -\epsilon_p, 0] \\ 0 & \mathcal{G}_1 & 0 & \dots & 0 \\ 0 & 0 & \mathcal{G}_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & 0 & \mathcal{G}_p \end{bmatrix}.$$

Above,  $\epsilon_s$  is a fixed sign associated with edge  $e_s$ , i.e., we have  $\epsilon_s^2 = 1$ . We are solving  $\mathcal{G}\mathbf{u} = \mathbf{f}$ . Let the edges connecting the parent vertex  $i_0$  with the subtrees  $T_s$  be  $e_s = (i_0, i_s)$ ,  $s = 1, \dots, p$ , and the respective values of  $\mathbf{f}$  be  $f_{e_s}$ . Also, the restrictions of  $\mathbf{f}$  to the edges of  $T_s$  be  $\mathbf{f}_s$ . Then, to solve  $\mathcal{G}\mathbf{u} = \mathbf{f}$ , we proceed as follows. Let  $\mathbf{u}_s + c_s \mathbb{1}_s$  be the solutions of  $\mathcal{G}_s(\mathbf{u}_s + c_s \mathbb{1}_s) = \mathcal{G}_s \mathbf{u}_s = \mathbf{f}_s$ , where  $c_s$  are arbitrary constants. Then, we can use these constants to satisfy the remaining  $p$  equations  $\epsilon_s(u_{i_0} - u_{i_s} - c_s) = f_{e_s}$ ,  $e_s = (i_0, i_s)$ ,  $s = 1, \dots, p$ . Here  $u_{i_s} = \mathbf{u}_s|_{i_s}$  (where  $i_s$  is the vertex in  $T_s$  that connects the parent one  $i_0$  with  $T_s$  via the edge  $e_s = (i_0, i_s)$ ). It is clear that one of these constants remains free, and to determine a unique overall solution  $\mathbf{u}$ , we can choose that free constant from the orthogonality condition  $(\mathbf{u}, \mathbb{1}) = 0$ . This shows, that  $\mathcal{G}$  is uniquely invertible on the subspace orthogonal to constants and in our case of tree graph, the cost is linear with respect to the number of edges of  $T$ .

Now, consider the adjoint problem  $\mathcal{G}^T \mathbf{u} = \mathbf{f}$ . Note that here  $\mathbf{f} = (f_i)$  is a vector defined on the vertices of  $T$  and  $\mathbf{u} = (u_e)$  is defined on the edges of  $T$ . From the equation

$$-\epsilon_s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_{e_s} + \mathcal{G}_s^T \mathbf{u}_s = \mathbf{f}_s,$$

using the fact that

$$(\mathcal{G}_s^T \mathbf{u}_s, \mathbb{1}_s) = (\mathbf{u}_s, \text{Grad}_{T_s} \mathbb{1}_s) = 0,$$

we obtain

$$(6.3) \quad -\epsilon_s u_{e_s} = (\mathbf{f}_s, \mathbb{1}_s).$$

We also have the equation for the parent vertex  $i_0$ ,

$$\sum_{s=1}^p \epsilon_s u_{e_s} = f_{i_0}.$$

This gives

$$-\sum_{s=1}^p (\mathbf{f}_s, \mathbb{1}_s) = f_{i_0},$$

which holds since  $\mathbf{f}$  is orthogonal to constants, i.e.,

$$0 = (\mathbf{f}, \mathbb{1}) = \sum_{s=1}^p (\mathbf{f}_s, \mathbb{1}_s) + f_{i_0} = 0.$$

Then the solvability of  $\mathcal{G}^T \mathbf{u} = \mathbf{f}$  follows by induction. We determine  $\mathbf{u}_s$  from the equation

$$(6.4) \quad \mathcal{G}_s^T \mathbf{u}_s = \mathbf{f}_s + c_s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The constant  $c_s$  is chosen such that  $\mathbf{f}_s + c_s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is orthogonal to  $\mathbb{1}_s$  which ensures, by induction, the solvability (in linear time) of (6.4). We have  $c_s = -(\mathbf{f}_s, \mathbb{1}_s)$ . Then,  $u_{e_s} = -\epsilon_s c_s$  (see (6.3)), which completes the proof (including the fact about the linear cost).  $\square$

The two-level Schwarz method provides an iterate  $\mathbf{u}_k$ , and the corresponding  $p_k \in S$  we obtain by solving

$$\text{Grad}_T p_k = \tilde{\mathbf{u}}_{k,T},$$

where  $T$  is a fixed spanning tree of  $G$ , same as in equation (6.2). Here,  $\tilde{\mathbf{u}}_{k,T}$  equals  $\mathbf{u}_k$  restricted to the tree edges. Recall that the problem for  $p_k$  above is solvable in linear time (by Proposition 6.3) because the graph  $T$  we consider is a tree.

$n$	$n_c$	$n_{\mathcal{E}}$	$n_{\mathcal{E}_c}$	$n/n_c$	$\rho$	$N_{it}$
250	29	688	69	8.6	0.57	25
494	49	1412	123	10.1	0.58	26
1016	106	2926	282	9.6	0.55	23
2040	207	5979	570	9.9	0.62	30
4156	420	12225	1189	9.9	0.63	30
8362	816	24803	2343	10.2	0.64	32
16924	1630	50287	4753	10.4	0.62	30
34022	3206	101499	9409	10.6	0.63	31

Table 6.1: Two-level Schwarz method (FE mesh example): one step of aggregation for constructing the coarse graph.

In Tables 6.1–6.3 we show the coarsening ratio  $n/n_c$ , the number of iterations  $N_{it}$ , and the average convergence rates  $\rho = \left(\frac{r_k}{r_0}\right)^{1/N_{it}}$  with  $r_k = \|\text{Div Grad } p_k + f\|_{\ell_2}/\|f\|_{\ell_2}$ . In the tables below we have also displayed the number of degrees of vertices  $n$  on the fine grid, the number of coarse grid vertices  $n_c$ , the number of edges in the fine graph  $n_{\mathcal{E}} = |\mathcal{E}|$  and the number of coarse grid edges  $n_{\mathcal{E}_c} = |\mathcal{E}_c|$  and In Table 6.1 and Table 6.2 the results for the planar (FE mesh) graph are shown. We test the two-level Schwarz method with one and two recursive aggregation steps and also for several levels of refinement. The graph is refined using the longest edge bisection algorithm.

In Table 6.3 we show the results for the two-level Schwarz method when applied to the graph from <http://opte.org>, for varying number of recursive aggregation steps (coarsening ratios ranging from  $\approx 3.5$  to  $\approx 740$ ). The results shown in Tables 6.1–6.3 show that the two-level Schwarz method presented converges and the convergence rate depends (as expected) on the coarsening ratio  $n/n_c$ .

## 7. CONCLUSIONS

In the present paper we have extended the notion of commuting projections from [1] exploiting the discrete divergence operator which is defined on general (connected) graphs. More specifically, based on commonly used aggregation-based coarsening, we define a coarse graph with vertices being the set of aggregates. A natural coarse subspace for the original vertex-based vector space is the space of piecewise constants with respect to the aggregates. We have then constructed a coarse subspace of the original edge-based vector space that complements the coarse piecewise constant vertex-based space. Based on the coarse edge-based space, we have constructed a computable projection, and we have proved a main commutative property (an analog of the result in [1]) for that projection and the standard  $\ell_2$ -based projection on the space of piecewise constants. Finally, we have presented numerical illustration of the use of the pair of the thus constructed coarse spaces in a two-level Schwarz

$n$	$n_c$	$n_{\mathcal{E}}$	$n_{\mathcal{E}_c}$	$n/n_c$	$\rho$	$N_{it}$
250	4	688	5	62.5	0.71	42
494	6	1412	9	82.3	0.78	56
1016	14	2926	29	72.6	0.73	45
2040	24	5979	55	85.0	0.78	55
4156	49	12225	124	84.8	0.83	75
8362	87	24803	226	96.1	0.83	73
16924	170	50287	467	99.6	0.84	80
34022	331	101499	925	102.8	0.84	83

Table 6.2: Two-level Schwarz method (FE mesh example): two steps of recursive aggregation for constructing the coarse space.

$n_c$	$n_{\mathcal{E}_c}$	$n/n_c$	$\rho$	$N_{it}$
9898	13409	3.6	0.84	82
2914	4809	12.2	0.88	112
815	1613	43.7	0.75	49
220	620	162.0	0.88	111

Table 6.3: Two-level Schwarz method: the graph is from <http://opte.org>. Convergence rates for varying coarsening ratio ( increasing number of recursive aggregation steps) are presented. In this example the number of vertices in the graph is  $n = 35638$  and the number of edges is  $n_{\mathcal{E}} = 42827$ .

method for solving the graph Laplacian problem in a mixed setting that exploits the discrete divergence operator. The edge-based quantities (in addition to the vertex-based ones) that we have touched upon in the present paper, provide us with an additional mathematical structure for developing analytical tools that might prove useful in formulating and analyzing problems on general graphs.

## REFERENCES

- [1] J. E. Pasciak and P. S. Vassilevski. Exact de Rham sequences of spaces defined on macro-elements in two and three spatial dimensions. *SIAM Journal on Scientific Computing*, 30(5):2427–2446, 2008.
- [2] Józef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Transactions of the American Mathematical Society*, 284:787–794, 1984.
- [3] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

- [4] Alan Gibbons. *Algorithmic graph theory*. Cambridge University Press, Cambridge, 1985.
- [5] Mikhail Shashkov. *Conservative finite-difference methods on general grids*. Symbolic and Numeric Computation Series. CRC Press, Boca Raton, FL, 1996. With 1 IBM-PC floppy disk (3.5 inch; HD).
- [6] A. Brandt, S. McCormick, and J. Ruge. Algebraic multigrid (AMG) for sparse matrix equations. In *Sparsity and its applications (Loughborough, 1983)*, pages 257–284. Cambridge University Press, Cambridge, 1985.
- [7] HwanHo Kim, Jinchao Xu, and Ludmil Zikatanov. A multigrid method based on graph matching for convection-diffusion equations. *Numerical Linear Algebra with Applications*, 10(1-2):181–195, 2003. Dedicated to the 60th birthday of Raytcho Lazarov.
- [8] Oren E. Livne and Achi Brandt. Lean Algebraic Multigrid (LAMG): Fast Graph Laplacian Linear Solver. submitted to SIAM Journal on Scientific Computing (SISC), August 2011, arXiv:1108.1310v1.
- [9] James J. Brannick, Yao Chen, and Ludmil T. Zikatanov. An Algebraic Multilevel Method for Anisotropic Elliptic Equations based on Subgraph Matching. *Numerical Linear Algebra with Applications*, pages 279–295, 2012.
- [10] P. S. Vassilevski. *Multilevel block factorization preconditioners*. Springer, New York, 2008. Matrix-based analysis and algorithms for solving finite element equations.
- [11] P. Vaněk, J. Mandel, and M. Brezina. Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems. *Computing*, 56(3):179–196, 1996. International GAMM-Workshop on Multi-level Methods (Meisdorf, 1994).

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